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COMMENT

Loop-erased self-avoiding random walk and the Laplacian random walk

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Abstract. We comment that the Laplacian random walk with $\eta = 1$, recently introduced by Lyklema and Evertsz, is the same process as the loop-erased self-avoiding random walk analysed previously by the author. Some rigorous results about this process are reviewed.

Attempts to understand the nature of the excluded volume effect on random walks have produced a number of measures on self-avoiding walks (SAW) which differ from the usual counting or uniform measure where every possible walk of a fixed length is given the same statistical weight. One major difficulty with the counting measure, which is the natural measure from the viewpoint of equilibrium statistical mechanics, is that it is not a kinetically growing model, i.e. the measure on $(n+1)$-step walks does not come from a conditional measure on n-step walks. A number of kinetically growing models have been introduced, none of which are exactly the same as the usual SAW, and it is of interest to ask how significant the difference is. In particular, are the walks in the same universality class as the usual SAW and, if not, do they at least have the same critical dimension and are the exponents in other dimensions close?

The author (Lawler 1980) introduced one of the first new measures by considering a measure induced on SAW by erasing loops from the paths of simple (unrestricted) random walks. It was shown that this process could also be considered as a kinetically growing random walk with a certain non-Markov transition probability. For this process a number of mathematically rigorous results have been obtained (Lawler 1980, 1986): the critical dimension is four (as is expected for the usual SAW); the process has long-range Gaussian behaviour for $d > 4$; and, for $d = 4$, there is Gaussian behaviour with a logarithmic correction term. In recent articles this process has been reintroduced by Lyklema and Evertsz (1986a, b) and Lyklema *et al* (1986) under the name of the Laplacian random walk with $\eta = 1$ (the latter authors actually define a one-parameter family of walks depending on η). Because it is not obvious that the processes are the same, we will sketch the argument from Lawler (1980), using some of the notation of Lyklema and Evertsz, that shows the equivalence. We then summarise some of the rigorous results about the model.

We start with a quick definition of the Laplacian random walk as defined by Lyklema and Evertsz. Consider the integer lattice Z^d and let R be a (large) number. If $d > 2$, R may be chosen to be infinity; for $d = 2$, R must be finite. We define a

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growing self-avoiding random walk by specifying the transition probability which tells how to get an $(n+1)$-step walk from an n-step walk. We assume $n+1<R$. Assume an n-step walk is given, say $\omega=[\omega(0)=0,\ldots,\omega(n)]$, where $\omega(i)$ denotes the position of the walk after i steps. Let $\Phi_R(x)$ be the function on Z^d satisfying

$$\Phi_R(x)=0 \qquad x\in\{\omega(0),\ldots,\omega(n)\} \tag{1a}$$

$$\Phi_R(x)=1 \qquad \max_i|x_i|\geq R \tag{1b}$$

where $x=(x_1,\ldots,x_d)$; and, for all other x,

$$\Phi_R(x)=\frac{1}{2d}\sum_{|e|=1}\Phi_R(x+e). \tag{1c}$$

That is, Φ_R is the function, harmonic with respect to the discrete Laplacian, with boundary conditions 0 on the path and 1 for x outside the box of size R. (If $d>2$ and $R=\infty$ the second boundary condition is 1 at ∞.) Then the Laplacian random walk with parameter η is the walk with transitions

$$P\{\omega(n+1)=y|\omega(0),\ldots,\omega(n)\}=(\Phi_R(y))^\eta\left(\frac{1}{2d}\sum_{|e|=1}(\Phi_R(x+e))^\eta\right)^{-1}. \tag{2}$$

In the above, $x=\omega(n)$ and the above holds for $|x-y|=1$. By standard facts about simple random walks (see, e.g., Spitzer 1976), the solution to $(1a)$-$(1c)$ is

$$\Phi_R(x)=\text{probability that a simple random walk starting at } x \text{ hits the boundary of the box of size } R \text{ before hitting } \{\omega(0),\ldots,\omega(n)\} \tag{3}$$

or, if $d>2$,

$$\Phi(x)=\Phi_\infty(x)$$

$$=\text{probability that a simple random walk starting at } x \text{ never enters } \{\omega(0),\ldots,\omega(n)\}.$$

It is easy to show for $d>2$ that $\Phi(x)=\lim_{R\to\infty}\Phi_R(x)$. It is not so obvious, but follows from a theorem of Kesten and Spitzer (1963), that for $d=2$ we can define $\Phi(x)=\Phi_\infty(x)$ by

$$\Phi(x)=\lim_{R\to\infty}\Phi_R(x)(\log R)$$

and use this for the transitions. Hence for $d=2$ we can choose $R=\infty$ if we mean it in the above sense.

We now define the loop-erased walk and sketch the argument to show that it is the same as the Laplacian random walk with $\eta=1$. Section 3 of Lawler (1980) gives the argument for $d>2$, $R=\infty$; for completeness we will perform here the case $R<\infty$. Essentially what we will do is take a simple random walk and erase the loops as illustrated in figure 1. To make this precise, let $R<\infty$ and $n<R$ be given. We define a measure on n-step SAW as follows: consider simple random walks starting at 0 and ending when they hit the boundary of the box of size R, i.e. at the first point $x=(x_1,\ldots,x_d)$ with $\max_i|x_i|\geq R$. Note that the walk must take at least R steps to get to the boundary. We will denote such a simple walk by $\xi=[\xi(0),\ldots,\xi(J)]$, reserving ω for self-avoiding walks. For such a simple walk ξ with $\xi(0)=0$,

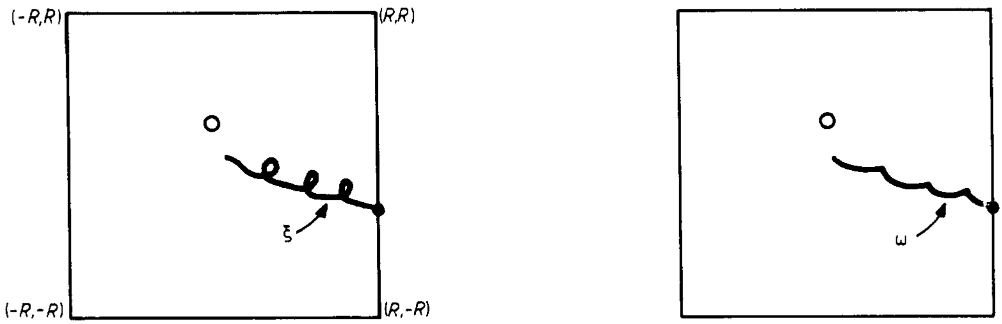

Figure 1. Illustration of loop-erasing procedure for $d=2$ and $R<\infty$.

$\max_i |\xi_i(J)| \geq R$, $\max_i |\xi_i(j)| < R$, $j=0,\ldots,J-1$, we will assign a SAW as follows. If ξ is self-avoiding we assign itself, otherwise let

$$\tau = \inf\{j>0\colon \xi(j)=\xi(i) \text{ for some } 0\leq i<j\}$$

$$\sigma = \text{the } i \text{ for which } \xi(i)=\xi(\tau).$$

Then send ξ to the $\bar{J}=(J-(\tau-\sigma))$-step path $\bar{\xi}$ given by

$$\bar{\xi}(i)=\begin{cases}\xi(i) & 0\leq i\leq\sigma\\ \xi(i+(\tau-\sigma)) & \sigma\leq i\leq\bar{J}.\end{cases}$$

Note that $\bar{\xi}$ is a $\bar{J}$-step walk whose first exit from the box of size R is at the $\bar{J}$ step (and hence $\bar{J}\geq R$). If $\bar{\xi}$ is self-avoiding we stop; otherwise, we perform this process on $\bar{\xi}$. Eventually, we will obtain a self-avoiding walk ω of length at least R. We assign $\xi\mapsto\omega$, and we have the measure on SAW given by

$$\hat{P}_R(\omega)=P\{\xi:\xi\mapsto\omega\}$$

where P denotes the probability measure on simple random walks. By considering only the first n steps of each ω we have a measure $\hat{P}_{R,n}$ on n-step SAW.

What we will show is that $\hat{P}_{R,n}$ is the same measure as that given by the transition probability (2) with $\eta=1$. We first note that by the Markov property for simple random walks, if $\omega_n=[\omega(0),\ldots,\omega(n)]$ is an n-step SAW and Φ_R is the solution to (1a)-(1c), then

$$\frac{1}{2d}\sum_{|e|=1}\Phi_R(\omega(n)+e)$$

= probability that a simple random walk starting at $\omega(n)$ leaves the box of size R before hitting $\{\omega(0),\ldots,\omega(n)\}$ (not counting the 0th step).

If $\omega_{n+1}=[\omega(0),\ldots,\omega(n),\omega(n+1)]$ is an extension of ω_n what we need to show is that

$$\frac{\hat{P}_{R,n+1}(\omega_{n+1})}{\hat{P}_{R,n}(\omega_n)}=\Phi_R(\omega(n+1))\left(\frac{1}{2d}\sum_{|e|=1}\Phi_R(\omega(n)+e)\right)^{-1}. \tag{4}$$

By (3) and the previous remark we can see that the RHS above equals the fraction of walks starting at $\omega(n)$ and leaving the box of size R before hitting $\{\omega(0),\ldots,\omega(n)\}$ whose first step is to the point $\omega(n+1)$.

Suppose $\xi=[\xi(0),\ldots,\xi(J)]$ is a simple random walk which produces ω_n when the loops are erased. Let $D=\sup\{j: \xi(j)=\omega(n)\}$. Then by definition of the loop-erasing procedure one can easily see that

$$\xi(j)\notin\{\omega(0),\ldots,\omega(n)\} \qquad j>D. \tag{5}$$

However, given D and $[\xi(0),\ldots,\xi(D)]$, the only restriction on $\xi(j), j>D$, so that the part of the path before D is not erased, is given by (5). Hence, given $[\xi(0),\ldots,\xi(D)]$ which produces $[\omega(0),\ldots,\omega(n)]$ upon loop erasing, one can extend $\xi(j), j>D$, any way which avoids $\{\omega(0),\ldots,\omega(n)\}$; the probability that $\xi(D+1)=\omega(n+1)$ is then given by (4). Note, by the definition of D, that if $\xi(D+1)=\omega(n+1)$, then ξ will produce ω_{n+1} upon loop erasing. This gives the result.

While the transition probability (Laplacian random walk) viewpoint for the process is nicer from a conceptual point of view, it is the loop-erased characterisation of the walk which has allowed mathematically rigorous analysis of the model. This model is the only model for SAW for which the critical dimension is known rigorously and for which the behaviour at the critical dimension can be described (for the counting measure and the related Domb-Joyce model, rigorous results are known above the critical dimension four (see Brydges and Spencer 1985, Slade 1986)). In Lawler (1980), it was shown for $d>4$ that the loop-erased process has the same behaviour as the simple random walk. More precisely, if $\omega(n)$ denotes the position of the nth step, then $\langle|\omega(n)|^2\rangle\sim cn$ for some $c>0$ and $(cn)^{-1/2}\omega(n)$ approaches a Gaussian distribution. For $d=4$, it has been shown (Lawler 1986) that $\langle|\omega(n)|^2\rangle\sim b_n n$, where b_n is a logarithmic correction term, and again that $(b_n n)^{-1/2}\omega(n)$ approaches a Gaussian distribution. Rigorously it is known that b_n grows at least as fast as $(\log n)^{1/3}$ and no faster than $(\log n)^{1/2}$. A very convincing, although not completely rigorous, argument gives that b_n grows in fact like $(\log n)^{1/3}$.

For $d=2$ and 3 the long-range behaviour of $\langle|\omega(n)|^2\rangle$ is unknown, although it is expected that it grows like $n^{2\nu}$ for some $\nu>\frac{1}{2}$. Some numerical work of Lyklema and Evertsz, solving Laplace's equation exactly, suggests ν is near 0.8 for $d=2$, as compared to the expected $\nu=0.75$ for the usual SAW. This method allowed calculation only up to 18-step walks. We expect that use of the loop-erased characterisation of the process should allow much more accurate numerical work. (The calculations of Lyklema and Evertsz allow them to analyse the Laplacian walk for all values of η; however, the computer time needed for only a small number of extra steps is very large.)

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